Int. J. Multiphase Flow Vol. 17, No. 3, pp. 377–394, 1991
Printed in Great Britain.

0301-9322/91 $3.00 + 0.00

NON-UNIQUE HOLDUP AND PRESSURE DROP IN TWO-PHASE STRATIFIED INCLINED PIPE FLOW

M. J. LANDMAN
Applied Mechanics and Electromagnetics Group, BHP Melbourne Research Laboratories,
P.O. Box 264, Clayton, Vic. 3168, Australia

(Received 16 May 1990; in revised form 6 November 1990)

Abstract—The simplified model of stratified two-phase pipe flow proposed by Taitel & Dukler [*Int. J. Multiphase Flow* **2**, 591–595 (1976)] has previously been reported to predict non-unique values of liquid holdup in some upward inclined pipes. This paper shows that the multivaluedness is not an artifact of the approximations made in the model, as it is also predicted by the exactly solvable case of laminar flow in an inclined square duct. A physical explanation for multiple liquid holdups is given, and their occurrence is parameterized in terms of simple flow variables. The stability of the three possible equilibria is discussed. A separated flow model predicts that, in general, the flow with the lowest holdup is the most stable, the highest equilibrium is unstable and the intermediate equilibrium can be stable or unstable, potentially leading to hysteresis phenomenon in slightly inclined flows.

Key Words: two-phase inclined pipe flow, stratified flow, liquid holdup prediction, pressure drop prediction

1. INTRODUCTION

Taitel & Dukler (1976a, b) proposed a simplified model for the prediction of liquid holdup in two-phase pipe flow. The flow is assumed to be one-dimensional and stratified along the length of the pipe, which can be slightly inclined from the horizontal. They published curves for the relationship between the height of the liquid layer (related to the holdup) and the Lockhart–Martinelli parameter X (related to the ratio of liquid to gas superficial pressure drops). Not only is the holdup predicted from this relation of intrinsic interest, but it may also be used to calculate the pressure drop and, furthermore, criteria for flow pattern transitions away from the stratified flow. The main attractions of the Taitel–Dukler (TD) model as a predictive tool is that it is simple to implement, and the only use made of empirical correlations is for the friction factors of each phase.

The curves presented by Taitel & Dukler (1976b) were qualitatively incorrect for upward inclined pipes, however. Corrected plots of the TD holdup relation have been given by Taitel & Dukler (1986), Barnea (1987), Baker & Gravestock (1987) and Baker *et al.* (1988). Baker *et al.* point out that the TD model can predict non-unique values of the holdup for given phase fluxes, although it appears that this issue has been left unresolved and an *ad hoc* choice of the holdup is made if this situation occurs in practice (e.g. in inclined gas/condensate pipelines, which often operate in the stratified flow regime, and where the gas flow rate is much greater than the liquid flow rate).

The purpose of this paper is to resolve whether the existence of multiple holdup values is a physically relevant phenomenon. Several aspects of this problem are discussed as follows.

In the next section, the TD holdup relation is reviewed for stratified one-dimensional two-phase flow in both circular pipes and rectangular ducts, with particular emphasis on the multivalued regime for upward inclinations. In section 3 the equivalent curves for the liquid holdup are derived from the two-dimensional fluid equations for an inclined rectangular duct. Although laminar duct flow is of limited relevance to most applications in fluid transport, this case is exactly solvable and provides a prototype for two-phase pipe flows. Direct comparison with the TD curves for one-dimensional laminar duct flows shows good qualitative agreement; in particular the non-uniqueness problem persists in the exact case. Section 4 gives a physical explanation for the multiple flow configurations, by observing detailed velocity profiles in laminar duct flow, in conjunction with momentum balance considerations.

Section 5 quantifies the criteria when multiple holdups occur, both by parameter fitting and asymptotic analysis. The findings are applied to specific pipelines, illustrating in superficial velocity coordinates the flow regime where non-uniqueness is predicted.

The stability of stratified flow in the multivalued regime is discussed in section 6, based on a dynamic one-dimensional separated flow model. Although the detailed analysis of the equations will be published elsewhere (Landman 1991), the results are summarized and corroborate the findings of Baker & Gravestock (1987) based on field-test data. A short conclusion follows in section 7.

2. MULTIVALUEDNESS IN THE TD HOLDUP RELATION

The TD equation for the prediction of the liquid level in a pipe is derived by performing a momentum balance on each of the liquid and gas phases in equilibrium, so that

$$\nabla P + F_L = 0, \quad \nabla P + F_G = 0. \qquad [1]$$

The pressure gradient ∇P is balanced by the forces due to shear and interfacial stresses, and the gravity force for an inclined flow, i.e.

$$F_L = \frac{1}{A_L}(\tau_{WL} S_L - \tau_i S_i + \rho_L A_L g \sin\theta) \qquad [2a]$$

and

$$F_G = \frac{1}{A_G}(\tau_{WG} S_G + \tau_i S_i + \rho_G A_G g \sin\theta). \qquad [2b]$$

See figure 1 for the definition of the geometric parameters, for both a circular pipe and rectangular duct. A_j and S_j are the cross-sectional area and wall perimeter of each of the gas and liquid phases, respectively ($j = G, L$). S_i is the interfacial perimeter. θ is the angle of inclination of the pipe from the horizontal, which is taken as positive for small upward inclinations (the opposite sign convention was taken in the TD analysis). The wall shear stresses are written in the form

$$\tau_{WL} = \tfrac{1}{2} f_L \rho_L u_L^2, \quad \tau_{WG} = \tfrac{1}{2} f_G \rho_G u_G^2, \qquad [3]$$

with friction factors

$$f_L = C_L \mathrm{Re}_L^{-n}, \qquad f_G = C_G \mathrm{Re}_G^{-m}. \qquad [4]$$

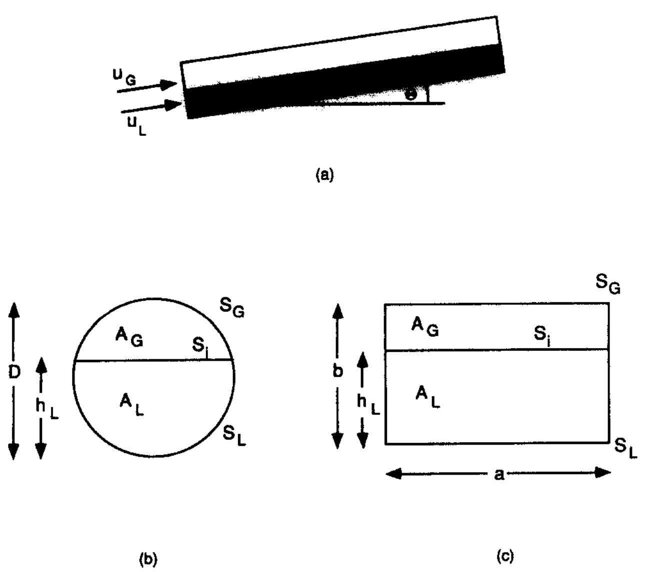

Figure 1. Geometry of stratified flow: (a) side view; (b) cross-section of a circular pipe; (c) cross-section of a rectangular duct.

u_G and u_L are the velocities of the phases, averaged across A_G and A_L. The actual Reynolds numbers $\mathrm{Re}_j = D_j u_j / \nu_j$ (j = G, L) are based on the hydraulic diameters

$$D_L = \frac{4A_L}{S_L}, \quad D_G = \frac{4A_G}{S_G + S_i}, \tag{5}$$

as introduced by Agrawal *et al.* (1973). In this way the situation is as if the liquid phase is in an open channel, and the gas phase is flowing in a closed duct. Furthermore, one of the assumptions made in the TD analysis that reduces the formulae to a very simple form is that the interfacial stress is the same as that for the gas phase at the wall, namely $\tau_i = \tau_{WG}$, which as above is based on the assumption that $u_L \ll u_G$. On non-dimensionalizing the length scales in the equations with respect to the diameter of the pipe, D (or the height b for the rectangular case), and the liquid and gas velocities with respect to each of their superficial values u_L^s and u_G^s, respectively, the TD holdup relation arises from [2a] and [2b]:

$$\alpha X^2 - \beta - 4Y = 0, \tag{6}$$

where

$$X^2 \equiv \frac{\left(\frac{dP}{dz}\right)_L^s}{\left(\frac{dP}{dz}\right)_G^s} = \frac{\frac{2C_L}{D}\left(\frac{u_L^s D}{\nu_L}\right)^{-n} \rho_L (u_L^s)^2}{\frac{2C_G}{D}\left(\frac{u_G^s D}{\nu_G}\right)^{-m} \rho_G (u_G^s)^2} \tag{7}$$

is the Lockhart–Martinelli parameter and

$$Y \equiv \frac{(\rho_L - \rho_G) g \sin\theta}{\left(\frac{dP}{dz}\right)_G^s} = \frac{-(\rho_L - \rho_G) g \sin\theta}{\frac{2C_G}{D}\left(\frac{u_G^s D}{\nu_G}\right)^{-m} \rho_G (u_G^s)^2} \tag{8}$$

is the TD inclination parameter. In the case of laminar flow with the friction coefficient C_f, these parameters simplify to

$$X^2 = \frac{\mu_L u_L^s}{\mu_G u_G^s}, \quad Y = -\frac{(\rho_L - \rho_G) g \sin\theta}{\frac{2C_f}{D^2} \mu_G u_G^s}. \tag{9}$$

α and β are purely geometrical parameters which are explicit functions of the liquid height, which in turn is a single-valued function of the holdup, i.e.

$$\alpha = (\tilde{u}_L \tilde{D}_L)^{-n} \tilde{u}_L^2 \frac{\tilde{S}_L}{\tilde{A}_L}$$

and

$$\beta = (\tilde{u}_G \tilde{D}_G)^{-m} \tilde{u}_G^2 \left(\frac{\tilde{S}_G}{\tilde{A}_G} + \frac{\tilde{S}_i}{\tilde{A}_L} + \frac{\tilde{S}_i}{\tilde{A}_G}\right), \tag{10}$$

where the tilde denotes the non-dimensional form of a geometrical parameter. For a circular pipe these functions are given by

$$\begin{aligned}
&\tilde{A}_G = \tfrac{1}{4}[\cos^{-1}(2\tilde{h}_L - 1) - (2\tilde{h}_L - 1)\sqrt{1 - (2\tilde{h}_L - 1)^2}], \\
&\tilde{A}_L = \tilde{A} - \tilde{A}_G, \quad \tilde{A} = \frac{\pi}{4}, \\
&\tilde{S}_G = \cos^{-1}(2\tilde{h}_L - 1), \quad \tilde{S}_L = \pi - \tilde{S}_G, \quad \tilde{S}_i = \sqrt{1 - (2\tilde{h}_L - 1)^2}, \\
&\tilde{u}_G = \frac{\tilde{A}}{\tilde{A}_G}, \quad \tilde{u}_L = \frac{\tilde{A}}{\tilde{A}_L}.
\end{aligned} \tag{11}$$

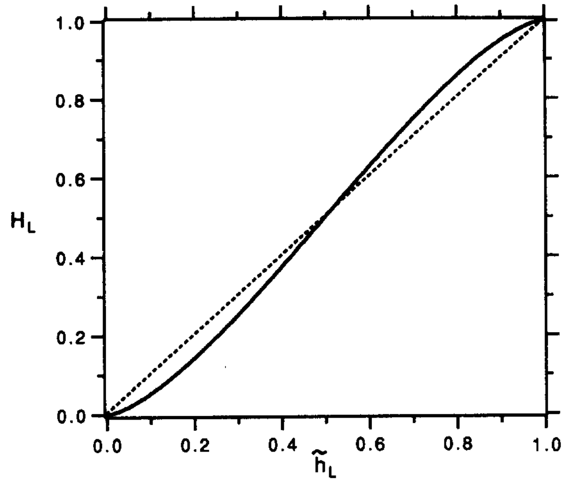

Figure 2. Liquid holdup vs non-dimensional liquid level for a circular pipe. The dashed line shows the linear relationship in a rectangular duct.

For a rectangular duct the TD relation still holds, but with the simpler geometrical factors:

$$\tilde{A}_G = \sigma(1 - \tilde{h}_L), \quad \tilde{A}_L = \sigma\tilde{h}_L, \quad \tilde{A} = \sigma,$$
$$\tilde{S}_G = \sigma + 2(1 - \tilde{h}_L), \quad \tilde{S}_L = \sigma + 2\tilde{h}_L, \quad \tilde{S}_i = \sigma,$$
$$\tilde{u}_G = \frac{1}{(1 - \tilde{h}_L)}, \quad \tilde{u}_L = \frac{1}{\tilde{h}_L}, \tag{12}$$

where $\sigma = a/b$ is the aspect ratio of the duct.

Note that $\tilde{u}_L$ is the reciprocal of the liquid holdup H_L in either geometry, which is really only a consequence of mass conservation. For the rectangular duct, the trivial relationship of $H_L = \tilde{h}_L$ holds, but for a circular pipe the holdup as a function of height is

$$H_L = 1 - \frac{1}{\pi}[\cos^{-1}(2\tilde{h}_L - 1) - (2\tilde{h}_L - 1)\sqrt{1 - (2\tilde{h}_L - 1)^2}], \tag{13}$$

which is shown in figure 2. The two quantites are numerically quite close, although the asymptotic behaviour of the curve for the circular pipe is non-linear in the limits $H_L \to 0$ or 1 (e.g. $H_L \sim c\tilde{h}_L^{3/2}$ as $\tilde{h}_L \to 0$). The asymptotic limits of the holdup relation are discussed further in section 5.

The pressure drop predicted by the TD model is found from [1] and [2b] for the momentum balance of the gas phase, and can be written in non-dimensional form as (Taitel & Dukler 1976a)

$$\frac{\left(\dfrac{dP}{dz}\right)}{\left(\dfrac{dP}{dz}\right)^s_G} = Q(\tilde{h}_L) - \frac{\rho_G}{\rho_L - \rho_G} Y, \tag{14}$$

where

$$Q(\tilde{h}_L) = (\tilde{u}_G \tilde{D}_G)^{-m} \tilde{u}_G^2 \tilde{D}_G^{-1}. \tag{15}$$

The function Q is a monotonically increasing function of liquid level, with a minimum $Q(0) = 1$.

The predictions of the TD holdup relation [6] for a circular pipe are now discussed. The friction factors required for its evaluation are taken as $C_L = C_G = 16$, $n = m = 1$ for laminar flow, and $C_L = C_G = 0.046$, $n = m = 0.2$ for turbulent flow. In practice the regimes for each phase would be determined from the actual Reynolds numbers Re_L and Re_G based on actual velocity and hydraulic diameter, with turbulent flow assumed for $Re > 2000$.

Figure 3 is a graph of the liquid level $\tilde{h}_L$ vs $\log X$ at various pipe inclinations in the turbulent gas/turbulent liquid regime. The graph differs from that in the original paper of Taitel & Dukler (1976b), where $\tilde{h}_L$ was incorrectly shown as being a single-valued function of $\log X$ at all inclinations. The relationship has since been shown correctly by Taitel & Dukler (1986), Barnea

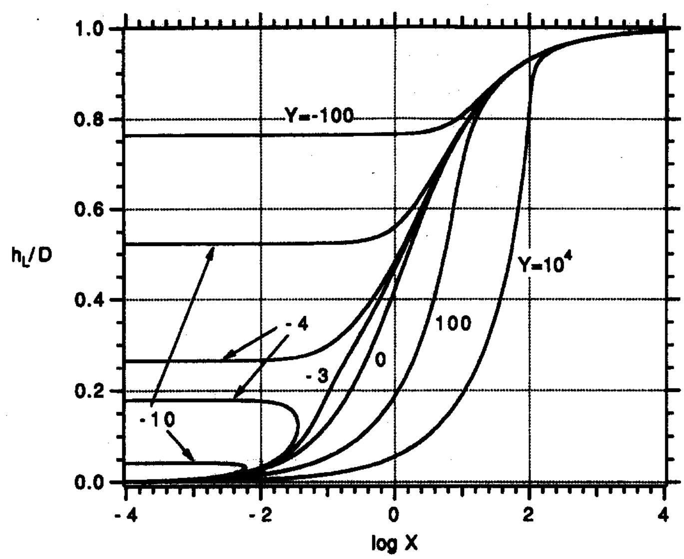

Figure 3. The TD relation showing liquid level vs the Lockhart–Martinelli flux parameter (circular pipe, turbulent/turbulent flow regime). The pipe is inclined upwards for negative values of the parameter Y.

(1987), Baker & Gravestock (1987) and Baker *et al.* (1988); in particular, the last two papers refer to the fact that the liquid level is triple-valued in an upwardly inclined pipe for a low ratio of liquid to gas flow (low X). This leads to a dilemma in determining which value of the holdup to choose for a class of inclined stratified flows for which the gas and liquid fluxes are specified. Baker *et al.* (1988) found that the holdup predicted by their existing codes based on the TD relation was sometimes far larger than observed in four of the five pipelines they studied. This anomaly could be explained by their algorithm producing an incorrect larger value for the holdup when the flow was in the multivalued regime. Their solution to this problem was to set the value of the parameter Y to a minimum value of -3.8 whenever its value was lower than this bound (this is approximately the limiting case for single-valuedness). This arbitrary procedure was chosen in the absence of their finding any physical significance in multivalued holdup curves.

The characteristics of the multivalued regime are shown in more detail in figure 4, for $Y = -5$. All flow regime combinations are shown, and verify the claim by Taitel & Dukler (1976a, b) that laminar and turbulent flows have similar holdup relations. The biggest deviations occur

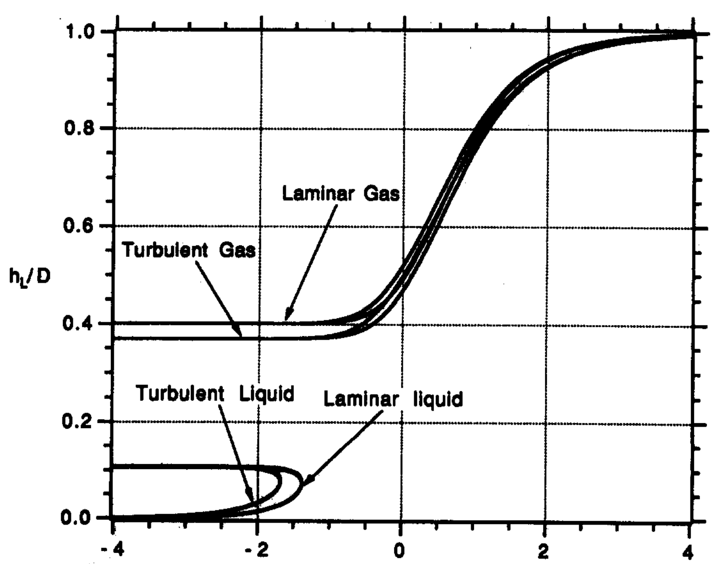

Figure 4. The TD relation for the four possible flow regimes, $Y = -5$.

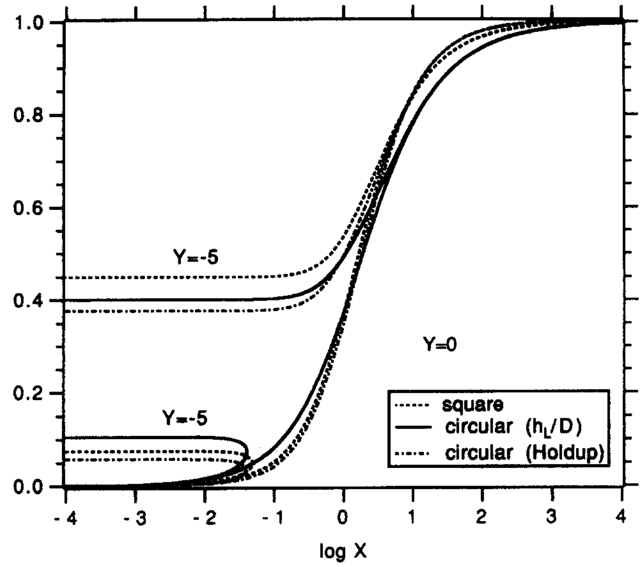

Figure 5. Comparison between laminar/laminar flow in a circular pipe and a square duct, for the TD model at $Y = 0$ and -5. Both the liquid level and holdup of the pipe flow are shown (these quantities are identical for the duct).

at low liquid flow rates: for low liquid holdup (low $\tilde{h}_L$) the gas regime does not affect the curves, and at higher holdup it is the gas flow regime that becomes more important, which is as might be expected.

In order to further verify the TD relation, the curves were calculated for a square duct (aspect ratio $\sigma = 1$), assuming laminar flow (friction coefficient $C_L = C_G = 14.22$ in [4], with $n = m = 1$). Figure 5 shows three curves for two values of the slope parameter ($Y = 0$ and -5); the first is for the square duct, for which the holdup and the liquid level are the same; the other two curves are for laminar flow in a circular pipe, plotting $\log X$ vs the liquid level $\tilde{h}_L$ and the holdup H_L (using [13] for the relationship between the latter two quantities, drawn in figure 2). There is good qualitative agreement of the square and circular geometries. Also, in general there is better quantitative agreement when the geometries are compared on the basis of holdup—this is desirable as the holdup is a more general measure of flow pattern than the liquid level.

An explanation of the numerical method used to produce the above graphs of $\tilde{h}_L$ vs X^2 is appropriate at this point. The TD relation [6] gives $X^2(Y, \tilde{h}_L)$ explicitly. Plotting X^2 over the range of $\tilde{h}_L$ for a fixed Y is the easiest method for getting the form of the holdup curves. An arc-length continuation method was used to solve [6], however, which is implicit (using Newton's method) but allows the accurate determination of any of the parameters as a function of the others along a solution branch. This is particularly useful in order to accurately locate the folds (turning points) in the holdup curves. The software package AUTO was used to carry out this procedure, as described by Doedel (1984).

Lastly, the non-dimensional pressure drop [14] can be calculated as a function of the phase fluxes, combining [6] and [15]. Figure 6 displays Q vs $\log X$ for slope $Y = -5$ and a turbulent gas/liquid regime. Figure 6 demonstrates that when the liquid flux is varied with the pipe diameter, inclination and gas flux fixed (giving $Y = -5$), the pressure drop can be triple-valued. This follows from the fact that Q is a single-valued function of the liquid level, which is simultaneously triple-valued.

Furthermore, for pipeline design purposes, if the pipe inclination and the fluxes of the two phases are specified, then an interesting relationship can arise between the actual pressure drop, liquid level and pipe diameter. In figure 7 these quantities are plotted for volumetric flow rates of gas/oil at 68 atm and 38°C of $q_G = 13.4\,\mathrm{m^3\,s^{-1}}$ and $q_L = 0.03\,\mathrm{m^3\,s^{-1}}$, at an inclination of 1°. For a fixed diameter, the holdup (liquid level) and pressure drop can be triple-valued, as was found above. Note, however, that if the pressure drop is specified, then the sizing of the pipe can also be non-unique (yielding two suitable values for the diameter). This added complication is because both

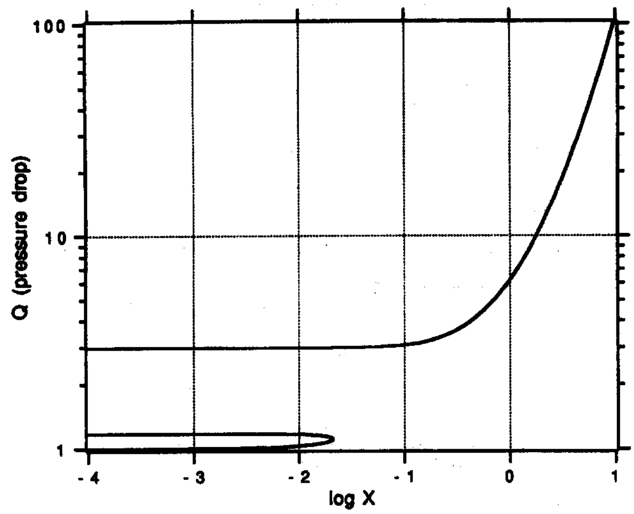

Figure 6. Non-dimensional pressure gradient vs the Lockhard–Martinelli flux parameter, when $Y = -5$, for the TD model in the turbulent/turbulent regime.

$(\mathrm{d}P/\mathrm{d}z)^s_G$ and Y are functions of the diameter D in [14], and also because keeping the volumetric flow rates constant implies that $D^2 u^s_j = \text{const}$ for each phase.

Up to this stage in the analysis, it is possible that the multivaluedness is an artifact of the simplifying assumptions on which the TD equation is based, and not the true underlying physics. In the next section, however, the existence of a multivalued holdup regime in an inclined pipe is confirmed by considering the full fluid equations for a duct, as distinct from the one-dimensional approximation made by Taitel & Dukler (1976a, b).

3. LAMINAR STRATIFIED DUCT FLOW

The equations describing stratified laminar two-phase flow in an inclined rectangular closed duct are considered. Similar equations were solved for a horizontal duct by Tang & Himmelblau (1963) and Charles & Lilleleht (1965), whose interest was pressure drop reduction in oil transport.

Taking a cartesian coordinate system parallel with the duct, inclined at an angle $\theta (\theta > 0$ for upward inclinations), the momentum equation for each phase is ($j = \mathrm{L}$ or G)

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u_j = f_j, \qquad [16]$$

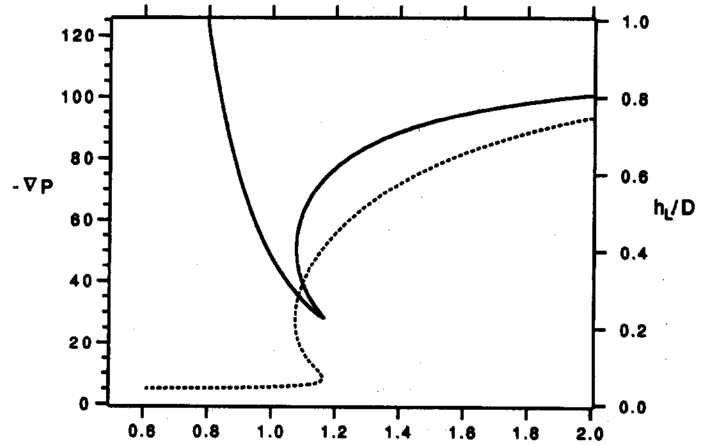

Figure 7. Actual pressure gradient ($\mathrm{kg\,m^{-2}\,s^{-2}}$) vs pipe diameter (m) for turbulent/turbulent flow for gas/oil at 68 atm and 38°C (——). The liquid level is shown by the dashed curve. The pipe is inclined 1° and the volumetric flow rates are fixed at $q_G = 13.4\,\mathrm{m^3\,s^{-1}}$ and $q_L = 0.03\,\mathrm{m^3\,s^{-1}}$.

where

$$f_j = \frac{1}{\mu_j}\left(\frac{\mathrm{d}P}{\mathrm{d}z} + \rho_j g \sin\theta\right). \quad [17]$$

The boundary conditions are that the velocity vanishes at the walls $x = (0, a)$ and $y = (0, b)$, and that the fluid velocity, pressure and the tangential stress are continuous across the plane interface located at $y = h_L$ [see figure 1c].

When the pipe has a circular cross-section, numerical means must, in general, be used to solve for the velocity field. In the rectangular geometry, however, a Fourier series solution can be generated, leading to a solution of the form [following Tang & Himmelblau (1963)]:

$$u_G = \sum_{n=1}^{\infty} [A_n \sinh(b_n - y_n) + B_n \cosh(y_n - h_n) + E_n] \sin x_n, \quad 0 \leqslant y \leqslant h_L, \quad [18]$$

where $y_n = n\pi y/a$, $x_n = n\pi x/a$, $b_n = n\pi b/a$ and $h_n = n\pi h_L/a$; and similarly for the liquid phase,

$$u_L = \sum_{n=1}^{\infty} [C_n \sinh y_n + D_n \cosh(y_n - h_n) + G_n] \sin x_n, \quad h_L \leqslant y \leqslant b. \quad [19]$$

The coefficients are found as functions of $f_{GL} = f_G/f_L$, liquid level $\tilde{h}_L = h_L/b$ and the viscosity ratio $\mu_{GL} = \mu_G/\mu_L$. It is found that (taking care that the terms in the resulting infinite series are numerically well-posed)

$$u_G = \frac{-4a^2 f_G}{\pi^3} \sum_{\substack{n=1 \\ n\,\mathrm{odd}}}^{\infty} \frac{\sin x_n}{n^3} \left\{1 + \frac{\gamma_n}{f_{GL}} \frac{\exp[-(y_n - h_n)] - \exp[-(2b_n - y_n - h_n)]}{1 + \exp[-2(b_n - h_n)]} - \frac{\exp[-(b_n - y_n)] + \exp[-(y_n + b_n - 2h_n)]}{1 + \exp[-2(b_n - h_n)]}\right\} \quad [20a]$$

and

$$u_L = \frac{-4a^2 f_L}{\pi^3} \sum_{\substack{n=1 \\ n\,\mathrm{odd}}}^{\infty} \frac{\sin x_n}{n^3} \left\{1 - \mu_{GL}\gamma_n \frac{\exp[-(h_n - y_n)] - \exp[-(y_n + h_n)]}{1 + \exp(-2h_n)} - \frac{\exp[-(2h_n - y_n)] + \exp(-y_n)}{1 + \exp(-2h_n)}\right\}, \quad [20b]$$

where

$$\gamma_n = \frac{[1 - \operatorname{sech} h_n] - f_{GL}[1 - \operatorname{sech}(b_n - h_n)]}{\mu_{GL} \tanh h_n + \tanh(b_n - h_n)}. \quad [21]$$

The fluxes (superficial velocities) are then found by integrating the velocity field across the duct, so that

$$u_G^s = \frac{-8a^3 f_G}{b\pi^5} \sum_{\substack{n=1 \\ n\,\mathrm{odd}}}^{\infty} \frac{1}{n^5} \left\{b_n - h_n - \tanh(b_n - h_n) + [1 - \operatorname{sech}(b_n - h_n)] \frac{\gamma_n}{f_{GL}}\right\} \quad [22a]$$

and

$$u_L^s = \frac{-8a^3 f_L}{b\pi^5} \sum_{\substack{n=1 \\ n\,\mathrm{odd}}}^{\infty} \frac{1}{n^5} [h_n - \tanh h_n - (1 - \operatorname{sech} h_n)\gamma_n \mu_{GL}]. \quad [22b]$$

For single-phase rectangular duct flow, the friction coefficient C_f can be calculated from

$$C_f = -\left(\frac{\mathrm{d}P}{\mathrm{d}z}\right) \frac{D_H^2}{2\mu u^s}, \quad [23]$$

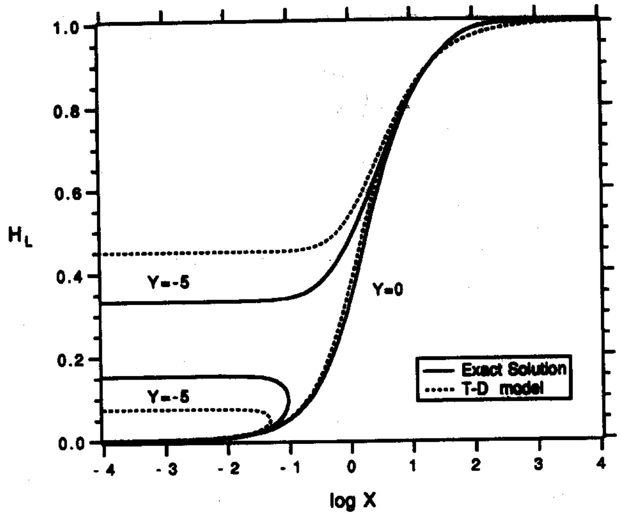

Figure 8. Holdup prediction in laminar square duct flow. The solid curve is the exact solution; the dashed curve is the TD model.

where $D_H = 2ab/(a+b)$ is the hydraulic diameter; and the superficial velocity u^s can be calculated from the liquid or gas phase above, in the limits $h \to 0$ or $h \to 1$, respectively. This gives the expression

$$C_f = \frac{24}{(\sigma+1)^2}\left(1 - \frac{192\sigma}{\pi^5}\sum_{\substack{n=1 \\ n\ \text{odd}}}^{\infty} \frac{\tanh\frac{b_n}{2}}{n^5}\right)^{-1}, \qquad [24]$$

where $\sigma = a/b$ is the aspect ratio and $b_n = n\pi/\sigma$, which was derived by Cornish (1928).

The Lockhart–Martinelli and slope parameters can now be evaluated exactly for a given flow, using definitions [9]. A comparison with the results of the exact analysis with the TD approximation for laminar two-phase flow in a rectangular duct can therefore be made. Figure 8 plots the holdup (or equivalently the liquid level) vs log X when the slope parameter $Y = 0$ and -5, using these two distinct methods of calculation. Good qualitative agreement is shown, in that multivaluedness is demonstrated for the upward sloping duct. The quantitative difference appears due to the approximate stresses used in the TD analysis, despite no approximations being made to the laminar friction factors. This difference suggests that improvement of the stresses used in the TD model could be made, which is an aspect of this study which the author hopes to pursue in the future. In particular, as shown in the next section, the liquid wall stress is of the wrong sign on the two upper branches of the holdup curve in the one-dimensional model.

It is important to note that the natural parameters X and Y that appear in the TD analysis allow the evaluation of the relationship $H_L(X, Y)$ to be carried out without explicit dependence on the physical properties of the fluids. In the exact analysis, however, the parameter dependence of the holdup is $H_L(X, Y, \rho_G/\rho_L, \mu_G/\mu_L)$, and as a result the curves in figure 8 were calculated for a specific pair of fluids (methane/oil at standard conditions), where $\rho_G = 0.72\ \text{kg m}^{-3}$, $\mu_G = 1.0 \times 10^{-5}\ \text{kg m}^{-1}\ \text{s}^{-1}$, $\rho_L = 900\ \text{kg m}^{-3}$ and $\mu_L = 1.0 \times 10^{-3}\ \text{kg m}^{-1}\ \text{s}^{-1}$. In addition to this pair of fluids, the holdup relation for gas/oil at 68 atm and for fluids of the same viscosity and different densities has been calculated, and the results appear to be quite insensitive to the viscosity and density ratios. This confirms that the quantities X and Y are the "correct" variables characterizing the flow. Once again the package AUTO was used to follow the solution branches in the above computations. The summations were carried out to at least 50 terms, with a convergence criterion of 6-figure accuracy.

4. PHYSICAL INTERPRETATION OF THE MULTIVALUED REGIME

It is now instructive to examine upwardly inclined laminar flows in more detail, in order to provide a physical interpretation for the occurrence of the multivalued phenomena.

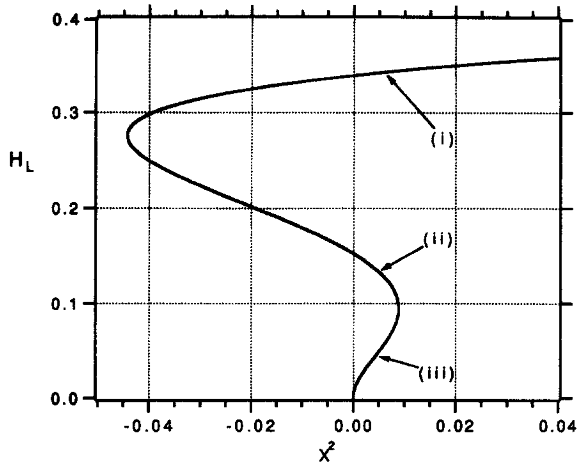

Figure 9. Holdup vs the Lockhart-Martinelli flux parameter, exact calculation.

First, in figure 9, the liquid holdup is plotted vs X^2 for exact square duct flow when $Y = -5$ (i.e. the Lockhart-Martinelli parameter is plotted directly, without taking a logarithm). The data in this form allows the description of net backflow when $X^2 < 0$ (note that negative values are allowed due to definition [9]). The fluids are taken as oil and natural gas at 68 atm and 38°C. ($\rho_G = 50\ \mathrm{kg\,m^{-3}}$, $\mu_G = 1.5 \times 10^{-5}\ \mathrm{kg\,m^{-1}\,s^{-1}}$, $\rho_L = 650\ \mathrm{kg\,m^{-3}}$, $\mu_L = 5.0 \times 10^{-4}\ \mathrm{kg\,m^{-1}\,s^{-1}}$).

The centreline flow profiles are plotted in figure 10 at the three equilibria indicated on the holdup curve in figure 9 at which $X^2 = 5.0 \times 10^{-3}$ and $Y = -5$ (i.e. in each case the externally imposed flow conditions are the same). Taking a duct of width 10 cm inclined upwards at 10°, the resulting

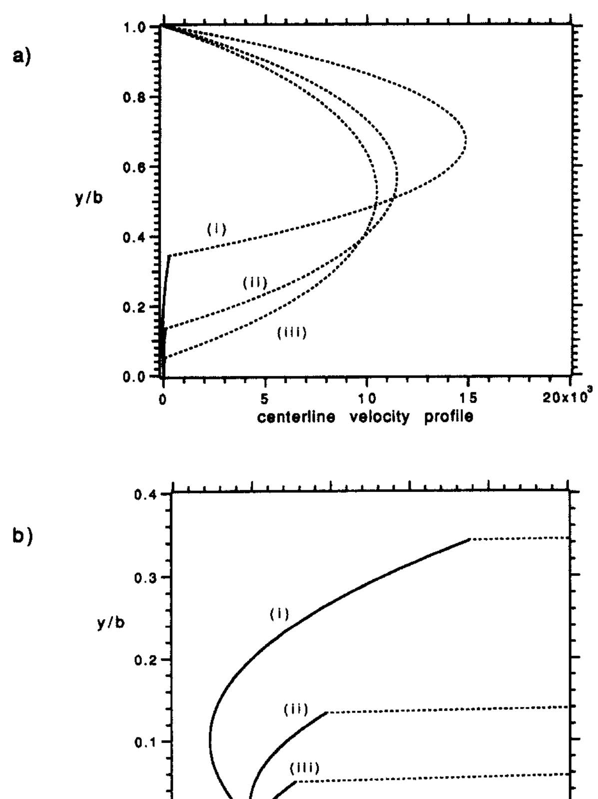

Figure 10. (a) Centreline velocity profiles for duct flow at the three points shown in figure 9: ——, liquid phase; ----, the gas phase. (b) Liquid flow in detail.

(negative) pressure gradients are (i) 591, (ii) 358 and (iii) 310 kg m^{-2} s^{-2} at the solutions with holdup (i) 0.342, (ii) 0.133 and (iii) 0.0488.

In discussing the form of the holdup curve, notice that for small H_L, X^2 increases from zero, which is reflected in the velocity profile (i), which is positive everywhere. The increase in flow rate with holdup is a property independent of the slope Y. This can be shown by determining the dominant balance in the one-dimensional force balance equations [1] and [2] for a small liquid level ($\tilde{h}_L = H_L$), where the dominant terms for the liquid phase are the interfacial and wall stresses, and not the gravitational terms (this is discussed in more detail in section 5). Similarly, for large H_L and large X^2 the relationship becomes independent of Y.

For intermediate values of H_L, however, the relationship is no longer monotonic once the pipe is upward sloping, due to the competition between the acceleration due to gravity and the axial pressure gradient. Taking $-Y$ sufficiently large and keeping the superficial gas velocity fixed, consider controlling the liquid level in the duct by varying the superficial liquid velocity (and thus the parameter X^2). As the liquid holdup is increased from zero, the large liquid wall and interfacial stresses will decrease and therefore the deceleration due to gravity will become more important, until it retards the liquid flux sufficiently that X^2 begins to decrease. This accounts for the lower fold in the holdup curves computed, and must always occur for $X^2 > 0$.

As the holdup is increased further, the gas wall stress and interfacial stress will increase due to the increasing velocity in the gas layer (recall the gas superficial velocity is considered fixed). This in turn accelerates the liquid phase, causing the liquid velocity to increase. At the point where this factor begins to dominate the deceleration due to gravity and liquid wall stress, the second fold in the holdup relation occurs, when the superficial liquid velocity begins to increase again. This point may occur when the superficial liquid velocity is of either sign, though for $-Y$ sufficiently greater than critical it occurs when $X^2 < 0$ (liquid flux negative), as is the case shown in figure 9.

Notice that along the middle branch of the holdup curve, the laminar flow profiles start to exhibit liquid back-flow, even though the liquid flux may be of either sign. The point at which back-flow begins (i.e. when the liquid wall stress changes sign) does not coincide with the lower fold in figure 9, although in the example shown it occurs soon after that point [i.e. between the lower fold and the point (ii) shown]. Persen (1987) also considered typical liquid velocity profiles in the uphill flow of liquid and gas, but failed to illustrate the case where there is no back-flow, which appears to be the most stable flow configuration (see section 6).

It is interesting that even though the liquid wall stress may become negative while the liquid flux remains positive, the TD one-dimensional model essentially remains valid. This is because the liquid wall stress no longer plays a significant role in the momentum balance. The incorrect sign of the TD wall stress was previously thought to invalidate one-dimensional inclined flow models (Shoham & Taitel 1984).

5. PARAMETERIZATION OF THE MULTIVALUED REGIME

A better understanding of the holdup curves can be achieved in terms of some simple concepts from bifurcation theory (Iooss & Joseph 1981). Each multivalued holdup curve has a pair of folds with coordinates (H_1, X_1^2) and (H_2, X_2^2), which depend on the given value of Y (as in figure 9). If Y is now varied, these folds can be followed computationally (this is easily implemented with the software AUTO). Figure 11 shows the result of such a fold continuation for the exactly solvable case of a square duct transporting methane/oil. On the curve of X^2 vs Y (solid line), a cusp is observed, which is the generic behaviour which occurs when two folds coalesce, according to bifurcation theory. The cusp occurs at the minimum value of $-Y$ for which the holdup relation is multivalued ($Y = -4.34$, $X^2 = 1.83 \times 10^{-2}$, $H_L = 0.172$, for laminar square duct flow). The values of (H_L, Y) at the folds is also plotted (dashed line) in figure 11.

Fold continuation has also been performed for the TD equations for a circular pipe, and the results are plotted for the turbulent/turbulent regime in figure 12. The generic behaviour is again revealed, with critical values at the cusp ($Y = -3.74$, $X^2 = 2.19 \times 10^{-3}$, $\tilde{h}_L = 0.174$). These plots are useful as they predict when multivaluedness of the liquid level and holdup will occur for all values of the physical parameters, given an assumption of the nature of the flow (i.e. laminar

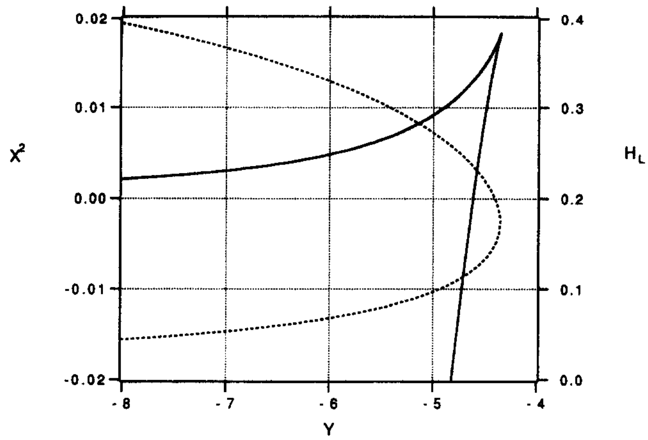

Figure 11. Locus of folds in the holdup curves for square duct flow: ——, (X^2, Y) coordinates of the folds; ----, (H_L, Y).

or turbulent, although the curves are fairly insensitive to these factors). Thus, the area bounded by the solid curve describes the region in X and Y coordinates for which multivalued holdup will occur. Bounds on the liquid level equilibria are also provided by the dashed curve in figure 12. Note that in the computations using the TD equation [6], not only is X^2 positive, which is the region of most practical interest, but X^2 has been allowed to become negative. This indicates a reversal in the liquid flux, requiring that the definition of the Lockhart–Martinelli parameter [7] be altered with the inclusion of appropriate absolute values (arising from the true stresses having the form $\tau = \frac{1}{2} f\rho u |u|$).

The region of multiple holdup predicted from the TD holdup relation when $X^2 > 0$ can be quantified by parameterizing the boundary of non-uniqueness in the X^2–Y plane (see figure 12):

(i) Right-hand boundary (coordinates of top fold):

$$Y = k_0 + k_1 X^2 + k_2 X^4, \quad 0 < X^2 \leqslant X_c^2; \qquad [25a]$$

(ii) Left-hand boundary (coordinates of bottom fold), intermediate values:

$$X^2 = X_c^2 + b_0 \tanh b_1(Y_c - Y) + b_2(Y_c - Y)\exp[-b_3(Y_c - Y)],$$
$$|Y_c| < |Y| \leqslant |Y_1|; \qquad [25b]$$

and

(iii) Left-hand boundary (coordinates of bottom fold), small X:

$$\log|Y| = a_0 + a_1 \log X, \quad |Y_1| < |Y|. \qquad [25c]$$

X_c and Y_c are the critical values at the cusp.

Table 1 displays the values of the constants fitted to these parameterizations, for both turbulent/turbulent flow (figure 12) and also turbulent gas/laminar liquid flow, when the results are qualitatively the same.

In addition to [25a–c], the asymptotic regime of [6], for small values of the holdup and X, can be analysed for a circular pipe (the asymptotic expressions are simpler in a rectangular duct). For fixed Y, the dominant terms in the TD equation [6] give

$$X^2 \sim \frac{\alpha}{\beta} \sim \frac{\alpha_0}{\beta_0} \tilde{h}_L^{(3-n/2)} \quad \text{as} \quad \tilde{h}_L \to 0, \qquad [26]$$

where from the definitions [10] and the geometrical relations [11]

$$\alpha \sim \alpha_0 \tilde{h}_L^{(n/2-4)}, \quad \beta \sim \beta_0 \tilde{h}_L^{-1}, \qquad [27]$$

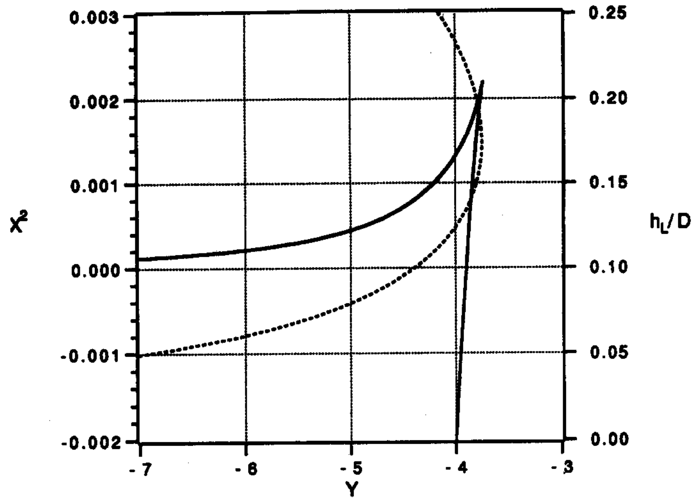

Figure 12. Locus of folds in the liquid level curves for the TD model of turbulent/turbulent flow: ——, (X^2, Y) coordinates of the folds; ----, ($\tilde{h}_L$, Y).

with

$$\alpha_0 = \frac{27\pi^2}{512}\left(\frac{\pi}{2}\right)^{-n}, \quad \beta_0 = \tfrac{3}{2}. \qquad [28]$$

These expressions give the asymptotic form of the TD curves at small holdup, using the relation for holdup in terms of the liquid level

$$H_L \sim \frac{16}{3\pi}\tilde{h}_L^{3/2}. \qquad [29]$$

Note that as described earlier, the dominant terms in [27] can be traced to the wall and interfacial stresses of the liquid phase.

Now the differentiation of the TD equation [6] with respect to the liquid level for fixed Y gives

$$\frac{d\alpha}{d\tilde{h}_L}X^2 + \frac{dX^2}{d\tilde{h}_L}\alpha - \frac{d\beta}{d\tilde{h}_L} = 0. \qquad [30]$$

At a fold, the second term vanishes, and thus

$$X^2 = \frac{\dfrac{d\beta}{d\tilde{h}_L}}{\dfrac{d\alpha}{d\tilde{h}_L}} \sim X_0^2\tilde{h}_L^{(3-n/2)} \quad \text{as} \quad \tilde{h}_L \to 0, \qquad [31]$$

Table 1. Constants parameterizing the region of multiple holdup

Constant	Turb. gas/Turb. liquid	Turb. gas/Lam. liquid
X_c	0.04678	0.08376
Y_c	−3.737	−3.697
Y_1	−17.80	−20.00
k_0	−3.896	−3.896
k_1	48.98	20.28
k_2	10550	1092
b_0	−0.002184	−0.006989
b_1	1.908	1.800
b_2	0.001264	0.004293
b_3	1.147	0.9744
a_0	−0.5851	−0.5259
a_1	−0.6895	−0.8000

with

$$X_0^2 = \frac{\beta_0}{\left(4 - \frac{n}{2}\right)\alpha_0}. \quad [32]$$

It is now possible to calculate the position of the lower fold as $\tilde{h}_L \to 0$. From the full TD equation

$$Y \sim \tfrac{1}{4}(\alpha X^2 - \beta) \sim \frac{3}{4}\left[\frac{1}{(8-n)} - \frac{1}{2}\right]\tilde{h}_L^{-1} = Y_0 \tilde{h}_L^{-1}, \quad [33]$$

so that eliminating $\tilde{h}_L$ from [31] and [33] gives

$$-Y \sim |Y_0 X_0^{4/(6-n)}| X^{-4/(6-n)}. \quad [34]$$

This accounts for the asymptotic form given in [25c], with the logarithm of the above constant of proportionality identified with a_0 and $a_1 = -4/(6-n)$. For a turbulent flow with $n = 0.2$, and for the laminar flow with $n = 1$, agreement with the above correlations is extremely good, providing a validation of the computational method.

The region where holdup non-uniqueness can occur can now be identified in terms of superficial velocity coordinates. The results are obtained from software that has been written that performs the following.

For a given diameter and pipeline inclination, at each point of an array in the u_G^s–u_L^s plane:

(a) The flow regime (laminar/turbulent for each phase) is guessed at.
(b) X^2 and Y are calculated.
(c) Test for $\tilde{h}_L$ multivaluedness; if single-valued, $\tilde{h}_L$ is calculated from the TD relation.
(d) The local Reynolds numbers are calculated; if these lie in a different flow regime than (a), start again but in the new flow regime.
(e) Flow pattern transition criteria (e.g. Taitel & Dukler 1976b) can be tested once steps (a)–(d) are consistent.

The method usually converges with respect to the flow regime. It is important to note that although step (c) is relatively insensitive to the choice of laminar or turbulent phases, step (b) can be quite sensitive to this choice, by virtue of [7] and [8] defining X and Y. This therefore can lead to occasional convergence problems at regime boundaries. Step (c) is carried out by two methods; the first by searching for multiple roots by evaluating $\tilde{h}_L$ over 25 subintervals of (0, 1), which is satisfactory if X^2 is not too small; the second method tests to see if the (X^2, Y) pair

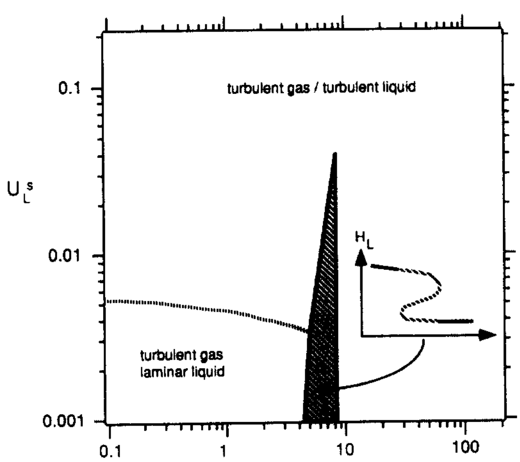

Figure 13. Flow map in superficial velocity coordinates, illustrating the region where multivalued holdup occurs. The inset graph shows the holdup as a function of u_G^s for fixed u_L^s across the region. The flow is of gas/oil at 68 atm, in a 25 cm pipeline inclined upward at 2°.

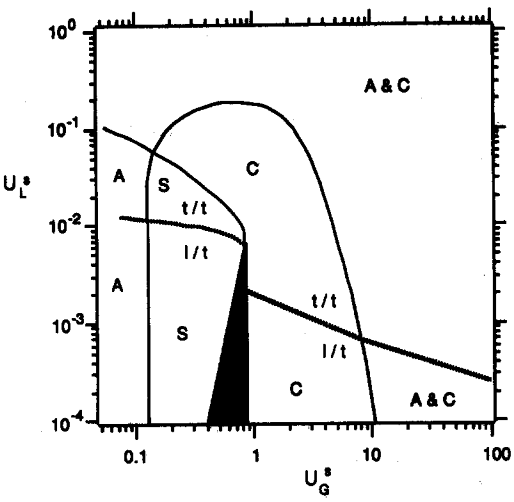

Figure 14. Flow map in superficial velocity coordinates, for gas/oil flow at 68 atm, in a 10 cm pipe inclined upwards at 0.1°. The map is derived from the TD holdup model, with gas/liquid flow regimes and predicted flow patterns shown: (C) wavy stratified; (A) intermittent/dispersed annular flow. A smooth stratified region (S) is predicted to exist, adjacent to the shaded region of non-unique holdup and pressure drop.

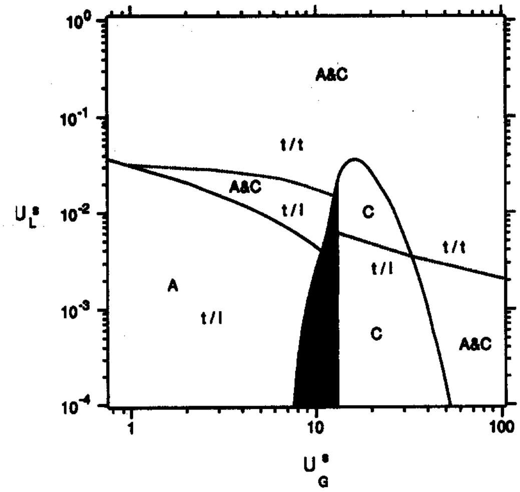

Figure 15. Flow map in superficial velocity coordinates, for air/water at 1 atm in a 5.1 cm pipe inclined at 1°. The region of stratified flow is predicted by the TD criteria to be wavy (C), in accordance with the experimental results (Barnea 1987). See the legend to figure 14 for a key to the other regimes.

lie in the multivalued region which was parameterized above. The choice of laminar or turbulent flow for the liquid phase only varies the multiple holdup boundaries slightly in the log–log coordinates; in the method used the regime is taken to be that of the neighbouring single-valued region.

As an example, figure 13 is plotted for stratified flow in a 25 cm pipe inclined upwards at 2°, for a high-pressure gas/oil flow. The region in which non-uniqueness is predicted is shaded. On the left of this region the holdup lies on the "upper branch" of the holdup curves (i.e. typically values $\gtrsim 0.5$), and to the right of the multivalued region the holdup lies on the "lower branch" (liquid holdup $\lesssim 0.1$).

In general, given liquid and gas flow rates, the flow pattern is not known *a priori*, and pattern transition must be accounted for. The code described can test the Taitel & Dukler (1976b) mechanistic criteria for transition from stratified flow, which are designated: (A) smooth stratified → intermittent/dispersal annular flow; and (C) smooth stratified → stratified wavy flow. The transition lines are very sensitive to pipe inclination, and the TD criteria exclude smooth stratified flow for the flow regime of figure 13. Figure 14 is drawn, however, for a 10 cm pipe inclined upwards at 0.1°, for a high-pressure gas/oil flow, when a smooth stratified region (S) is predicted to exist. The figure also shows where the TD criteria A and C have been satisfied. Furthermore, note that the non-uniqueness of the holdup is still predicted, and adjoins the smooth stratified region.

Finally, figure 15 shows the flow map predicted by the TD criteria for the case performed experimentally in air/water at 1 atm in a 5.1 cm pipe inclined upward at 1° (Barnea 1987). Stratified flow is predicted to exist only to one side of the multivalued region, in which case it is wavy (region C). The transition to non-stratified flow (region A) coincides very well with that found experimentally, as reported by Barnea (1987). Although the TD holdup model was strictly derived only for smooth stratified flow, this agreement supports the claim that it is still accurate if the interface is wavy, and thus that multiple holdups and pressure drops are possible within the wavy/stratified flow regime.

6. STABILITY OF MULTIPLE EQUILIBRIA

Having established the existence of multiple equilibria, it is important to determine which of the stratified flows is physically realizable. A comprehensive description of a stability analysis of the flows will be presented in the near future (Landman 1991). A summary of the approach taken and the results are given here.

A relatively simple but consistent approach to stability of stratified flow is to analyse the linear stability of a one-dimensional separated incompressible flow (Wallis 1969; Banerjee & Chan 1980). The conservation of mass and momentum equations for each phase are

$$\frac{\partial H_i}{\partial t} + \frac{\partial}{\partial x}[H_i u_i] = 0 \qquad [35a]$$

and

$$\frac{\partial u_i}{\partial t} + \frac{1}{2}\frac{\partial u_i^2}{\partial x} + gD\cos\theta\frac{\partial \tilde{h}_L}{\partial x} = -\frac{1}{\rho_i}\left(F_i + \frac{\partial P}{\partial x}\right). \qquad [35b]$$

The notation is the same as used in section 2, with $i = G, L$ for the gas and liquid phases, $H_G = 1 - H_L$ and $\tilde{h}_L$ the non-dimensional interfacial height. The stress and gravity terms are included in F_L and F_G, given by [2]. Note that for steady uniform flow, the l.h.s.s of the equations vanish, and the original force balance equations [1] result, leading to the TD equilibrium flows.

The stability of a stratified equilibrium is analysed by linearizing the equations of motion, which can be reduced to a single second-order one-dimensional wave equation (Wallis 1969). Note that a spatially uniform model, where only the time derivatives remain on the l.h.s. of [35a] and [35b], is discarded. Such a model would be inconsistent with mass conservation, although the resulting ordinary differential equation leads to the simple result that the top and bottom equilibria on the holdup curve are stable and the intermediate value is unstable.

In the time- and space-varying model adopted here, an equilibrium flow is considered stable if this equation is well-posed (hyperbolic), and a simple algebraic stability criterion is satisfied. This criterion relies upon the numerical evaluation of certain partial derivatives of the friction terms F_i.

The results show that typically a continuous band (parameterized by holdup) of stratified solutions is stable, for fixed superficial liquid velocity. This band of stable solutions lies generally on the lower part of the holdup curve, with the upper branch unstable.

For the flow maps of the previous section, the results are:

(1) Figure 13 (oil/methane inclined at 2°)—the model predicts that only the equilibrium with the lowest holdup is stable and that this only occurs in part of the multivalued region.
(2) Figure 14 (oil/methane inclined at 0.1°)—the model predicts that two stable equilibria coexist, namely those with the lower values of holdup.
(3) Figure 15 (water/air inclined at 1°)—three situations are possible depending on the flow rates; namely, all equilibria are unstable, the lowest is stable or the lower pair are stable.

Although both the linear stability model and the TD criterion predict the existence of stable smooth stratified flow, the experimental evidence suggests that uphill flows are always wavy (Barnea 1987). Therefore, further work is required to test the validity of the stability method used here. The results presented here are not unreasonable, however, given that flows with higher holdup are more likely to be unstable, both because of interfacial shear instability and the existence of back-flow.

7. CONCLUSION

It has been shown theoretically that non-unique equilibrium values of the liquid holdup and pressure drop can be expected to occur in stratified two-phase flow in upward inclined pipes. Multivaluedness occurs at low liquid-to-gas flow rates, and in general requires only very slight inclination angle. Such flow regimes are of practical importance, particularly in gas-condensate pipelines (Baker & Gravestock 1987).

Stability calculations demonstrate that when three equilibria are predicted, the equilibrium with the lowest holdup (exhibiting the least pressure drop) is the most likely to be stable, with the upper equilibrium unstable and the intermediate equilibrium either unstable or stable. In the latter case, hysteresis between stratified flows with different holdups and pressure drops is predicted by the model. The intermediate and highest equilibria may be expected to exhibit liquid back-flow near the wall, although both the TD theory and the calculations performed here for laminar duct flow show that back-flow does not necessarily occur.

The stability of the lower equilibria is consistent with the observation of Baker & Gravestock (1987) that the upper branch solutions, which were the only ones displayed in Taitel & Dukler's original papers (1976a, b), grossly overpredict the liquid holdup in upward inclined gas-condensate pipelines.

Criticism of the accuracy of the TD holdup model has been raised (Spedding & Spence 1989). The comparison with experimental data that they present is statistically crude, however, and includes data for low gas flow rates. This is a situation where the assumptions of the TD model (e.g. $u_G \gg u_L$) are not met, and is a regime not relevant to this work.

It remains an open question as to whether the predictions made here occur in practice. In experimental work, it is quite possible that multiple equilibria have gone unnoticed if the investigators were not looking for such phenomena. In order to detect multiple holdups, the flow regime would have to be approached by varying the inlet fluxes to their desired values in different ways. In the work of Beggs & Brill (1973) in air/water flows, stratified flow was found to occur only for upward inclinations of $<3°$, for which they do not display data. It is noticed, however, that their results for prediction of holdup and pressure drop were least satisfactory at the lowest inclination angle studied of 5°.

Persen (1987) performed experiments to specifically find the holdup in upward inclined stratified flows. The experimental data presented is for glycerol and air in a 5 cm 2° inclined pipe; however the TD model predicts single-valued holdup for the flow rates reported. Persen also presents a stability model for stratified flow based on the steady flow equations, but it is felt that the absence of time dependence makes the model questionable.

Experiments are currently being proposed at BHP Melbourne Research Laboratories in order to search for the multiple steady states suggested by the present study.

The theoretical results presented depend on the existence of stratified flow; at significant flow rates, experimental evidence of air/water flow in small diameter pipes suggests that stratified flow only persists at small upward inclination angles (Barnea *et al.* 1985; Barnea 1987), when the interface is actually wavy. Nevertheless, the one-dimensional momentum equations on which the equilibrium predictions are based hold in any steady flow regime, given that the appropriate forms of stress factors are used. Therefore, non-uniqueness of the liquid holdup and pressure drop should also not be excluded within other flow patterns, especially in upward inclined pipes, due to the non-linear nature of two-phase flow.

Acknowledgements—The author would like to thank Drs A. A. Inayat-Hussain, B. P. Marett and C. C. Lakshmanan for their interest in this work, and for valuable discussions. Dr E. J. Doedel is thanked for the use of the AUTO software. BHP is also acknowledged for its support.

REFERENCES

AGRAWAL, S. S., GREGORY, G. A. & GOVIER, G. W. 1973 An analysis of horizontal stratified two-phase flow in pipes. *Can. J. chem. Engng* **51,** 280–286.

BAKER, A. & GRAVESTOCK, N. 1987 New correlations for predicting pressure loss and holdup in gas/condensate pipelines. In *Proc. 3rd Int. Conf. on Multiphase Flow*, The Hague, The Netherlands, pp. 417–435.

BAKER, A., NIELSEN, K. & GABB, A. 1988 Pressure loss, liquid holdup calculations developed. *Oil Gas J.* **March 14,** 55–59.

BANERJEE, S. & CHAN, A. M. C. 1980 Separated flow models—I. *Int. J. Multiphase Flow* **6,** 1–24.

BARNEA, D. 1987 A unified model for predicting flow-pattern transitions for the whole range of pipe inclinations. *Int. J. Multiphase Flow* **13,** 1–12.

BARNEA, D., SHOHAM, O., TAITEL, Y. & DUKLER, A. E. 1985 Gas–liquid flow in inclined tubes: flow pattern transitions for upward flow. *Chem. Engng Sci.* **40,** 131–136.

BEGGS, H. D. & BRILL, J. P. 1973 A study of two-phase flow in inclined pipes. *J. Petrol. Technol.* **13,** 607–617.

CHARLES, M. E. & LILLELEHT, L. U. 1965 Co-current stratified laminar flow of two immiscible liquids in a rectangular conduit. *Can. J. chem. Engng* **43,** 110–116.

CORNISH, R. J. 1982 Flow in a pipe of rectangular cross-section. *Proc. R. Soc.* **A120,** 691–700.

DOEDEL, E. J. 1984 A computer aided analysis of predator–prey models. *J. math. Biol.* **20,** 1–14.

IOOSS, G. & JOSEPH, D. D. 1981 *Elementary Stability and Bifurcation Theory*. Springer, New York.

LANDMAN, M. J. 1991 Hysteresis in simulations of two-phase stratified inclined pipe flow. *Proc. 5th Int. Conf. on Multiphase Production*, Cannes, France. Submitted.

PERSEN, L. N. 1987 Fundamental concepts of two-phase flow in upwards sloping pipelines. *Energy Prog.* **7,** 170–179.

SHOHAM, O. & TAITEL, Y. 1984 Stratified turbulent–turbulent gas–liquid flow in a horizontal and inclined pipes. *AIChE Jl* **30,** 377–385.

SPEDDING, P. L. & SPENCE, D. R. 1989 Prediction of holdup in two-phase flow. *Int. J. Engng Fluid Mech.* **2,** 109–118.

TAITEL, Y. & DUKLER, A. E. 1976a A theoretical approach to the Lockhart–Martinelli correlation for stratified flow. *Int. J. Multiphase Flow* **2,** 591–595.

TAITEL, Y. & DUKLER, A. E. 1976b A model for predicting flow regime transitions in horizontal and near horizontal gas–liquid flow. *AIChE Jl* **22,** 47–55.

TAITEL, Y. & DUKLER, A. E. 1986 Flow pattern transitions in gas–liquid flow: measurements and modelling. In *Multiphase Science and Technology*, Vol. 2. Hemisphere, Washington, D.C.

TANG, Y. P. & HIMMELBLAU, D. M. 1963 Velocity distribution of isothermal two-phase co-current laminar flow in a horizontal rectangular duct. *Chem. Engng Sci.* **18,** 143–148.

WALLIS, G. B. 1969 *One-dimensional Two Phase Flow*. McGraw–Hill, New York.